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The penetration into conductors of magnetic fields from moving charges

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Abstract. The magnetic field produced in a plane conductor by a non-relativistic line charge moving parallel to the face is examined on the basis of classical electromagnetic theory. It is discovered that, at a fixed speed less than the local speed of light, skin effect is not a significant feature and the magnetic intensity decays algebraically rather than exponentially with distance from the face of the conductor. The multiplying factor does, however, depend upon the conductivity and becomes zero for a perfect conductor. For Čerenkov radiation, on the other hand, the position is reversed; skin effect is dominant and the attenuation is exponential. This result may affect the interpretation of the Aharonov–Bohm effect which is currently under discussion.

1. Introduction

In experiments concerning the Aharonov–Bohm effect (Aharonov and Bohm 1959) electrons travel near to microsolenoids or magnetic whiskers and thereby induce certain fields. Usually the effects are not attributed to classical electromagnetic forces (Bayh 1962), primarily because it is thought that the magnetic fields cannot penetrate deep enough into a conductor. The analysis (Kasper 1966) is essentially based on the phenomenon of skin effect which clamps the fields into a neighbourhood of the surface of the conductor, at any rate when the frequency of excitation is high enough. However, for a moving particle many frequencies are present and some of them may be low enough to invalidate this conclusion. For this reason Boyer (1974) decided to re-examine the classical electromagnetic problem when the speed of the particle is very small and formed the opinion that the fields in the conductor, far from being confined to the surface, did in fact permeate for a very considerable distance. As a consequence he has suggested (Boyer 1973) that classical electromagnetic forces might be involved in the Aharonov–Bohm effect.

The analysis of Boyer assumes that the permittivity and permeability of the conductor are the same as those of the surrounding medium. On this basis he concludes that there is no skin effect in the conductor and that both the magnetic field and current density in the conductor are independent of the resistivity. It is desirable to know to what extent these conclusions rest on the assumptions. In other words, would changes in the permittivity and permeability or the velocity not being vanishingly small cause significant alterations to the predictions?

Accordingly we consider the problem of a non-relativistic charge moving parallel to a plane conductor with general material constants. To simplify the analysis we assume that the charge is distributed along a line moving perpendicular to itself. It is not expected that employing a line charge instead of a point charge will produce such a substantial effect as converting an exponentially diminishing field into one with algebraic decay.

Qualitative agreement with Boyer's conclusions is found in that skin effect is not the fundamental phenomenon and the diminution of the magnetic field is algebraic rather than exponential. There are, however, substantial differences in detail. The rate of attenuation is more rapid than that specified by Boyer and the magnetic intensity does depend upon the resistivity with the net effect that it will tend to be appreciably smaller than would be deduced from Boyer's analysis. The differences are not due to the change in material constants but stem from the speed of the charge not being infinitesimally low. If the speed is allowed to approach zero Boyer's results are recovered. In other words the low speed phenomena form a special limiting case.

If we consider the other extreme, of increasing speed, the algebraic behaviour continues while the speed of the electron is less than the speed of light of the medium in which it is progressing, even if it is faster than the speed of light in the conductor. When the charge is travelling at a speed in excess of the speeds of light in both media the situation is totally altered. Now skin effect is dominant and the magnetic field decreases exponentially in the conductor. To put it another way, Čerenkov radiation does not penetrate far into a conductor on non-relativistic classical theory.

Thus to achieve greater magnetic intensity in the conductor one should increase the speed of the charge so long as it remains less than the local speed of light. However, no attempt has been made to investigate whether there is an optimal speed, so this must be regarded as a rough rule applicable to low speeds but not necessarily working for the whole speed range.

The mathematical formulation of the problem is given in § 2 and an exact solution found in terms of a Fourier integral. The integral is evaluated asymptotically in § 3 for points of observation in the conductor which are well away from the interface, under the assumption that the charge is moving at less than the speed of light in either medium. The modifications that are necessary when the speed of the charge is (*a*) greater than the speed of light in the conductor but less than that in the other medium, (*b*) greater than both speeds of light, are discussed in § 4.

2. Formulation

Let a line charge parallel to the x axis be moving in the direction of the positive z axis with constant speed v in the plane $y = -b$ $(b > 0)$. Assume that the medium is non-conducting, homogeneous and isotropic. Let the permeability and permittivity be μ and ϵ respectively, SI units being used throughout. Then Maxwell's equations are:

$$\begin{aligned} &\operatorname{curl} \boldsymbol{E} + \mu \frac{\partial \boldsymbol{H}}{\partial t} = 0 \qquad && \epsilon \operatorname{div} \boldsymbol{E} = Q\delta(y+b)\delta(z-vt) \\ &\operatorname{curl} \boldsymbol{H} - \epsilon \frac{\partial \boldsymbol{E}}{\partial t} = Qv\delta(y+b)\delta(z-vt)\boldsymbol{k} \qquad && \operatorname{div} \boldsymbol{H} = 0 \end{aligned} \tag{1}$$

where Q is a measure of the charge on the line, $\boldsymbol{k}$ is a unit vector parallel to the z axis and $\delta(x)$ is the usual Dirac δ function.

The conductor occupies the space $y \geqslant 0$ and its permeability, permittivity and conductivity are μ_1, ϵ_1 and σ respectively. These quantities are assumed to be constant and frequency-independent. The charge density in such a conductor decays exponentially from its initial value in a manner independent of any excitation. There is, therefore, no loss of generality in assuming it to be zero. With this understanding, Maxwell's equations in the conductor are:

$$\operatorname{curl} \boldsymbol{E}+\mu_1\frac{\partial \boldsymbol{H}}{\partial t} = 0 \qquad \operatorname{div} \boldsymbol{E} = 0$$
$$\operatorname{curl} \boldsymbol{H}-\epsilon_1\frac{\partial \boldsymbol{E}}{\partial t} = \sigma\boldsymbol{E} \qquad \operatorname{div} \boldsymbol{H} = 0. \tag{2}$$

The line charge does not produce any electric field parallel to itself or any magnetic field in the (y, z) plane. Since the presence of the conductor does not cause the generation of any such fields we may assume that the components E_x, H_y and H_z are always zero. Also, all other field components may be taken as independent of x.

Denote by small letters the Fourier transforms of the field components with respect to time so that, for example,

$$E_y = \frac{1}{2\pi}\int_{-\infty}^{\infty} e_y\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega.$$

Then in view of the foregoing remarks equations (1) become

$$\frac{\partial e_z}{\partial y}-\frac{\partial e_y}{\partial z}+\mathrm{i}\omega\mu h_x = 0 \qquad \frac{\partial e_y}{\partial y}+\frac{\partial e_z}{\partial z} = \frac{Q}{\epsilon v}\delta(y+b)\,\mathrm{e}^{-\mathrm{i}\omega z/v}$$
$$\frac{\partial h_x}{\partial z} = \mathrm{i}\omega\epsilon e_y \qquad \frac{\partial h_x}{\partial y}+\mathrm{i}\omega\epsilon e_z = -Q\delta(y+b)\,\mathrm{e}^{-\mathrm{i}\omega z/v}.$$

A solution of these equations is provided by

$$h_x = u \qquad e_y = (1/\mathrm{i}\omega\epsilon)\partial u/\partial z$$
$$e_z = -(1/\mathrm{i}\omega\epsilon)[\partial u/\partial y+Q\delta(y+b)\,\mathrm{e}^{-\mathrm{i}\omega z/v}]$$

so long as

$$\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+\frac{\omega^2}{c^2}u+Q\delta'(y+b)\,\mathrm{e}^{-\mathrm{i}\omega z/v} = 0, \tag{3}$$

where $c = 1/(\mu\epsilon)^{1/2}$ is the speed of light in the medium.

The Fourier inversion theorem implies that

$$e_y(\omega) = \int_{-\infty}^{\infty} E_y\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t.$$

Because E_y is real,

$$e_y(-\omega) = \{e_y(\omega)\}^* \tag{4}$$

where the star indicates a complex conjugate. Thus values of e_y for negative ω can be deduced from those for positive ω via (4). Alternatively

$$E_y = \mathrm{Re}\frac{1}{\pi}\int_0^\infty e_y\, \mathrm{e}^{\mathrm{i}\omega t}\, \mathrm{d}\omega. \tag{5}$$

In any event negative values of ω can be excluded from the following discussion.

A particular solution of (3) is

$$u_0 = -\tfrac{1}{2}Q\,\mathrm{sgn}\,(y+b)\,\mathrm{e}^{-\omega(\mathrm{i}z+\beta|y+b|)/v}$$

where $\beta = (1-v^2/c^2)^{1/2}$, while sgn x is 1 if $x > 0$ and -1 if $x < 0$. With $v < c$ the constant β is chosen as positive. The solution u_0 corresponds to the field produced by the moving charge in the absence of the conductor.

To take account of the presence of the conductor we put $u = u_0 + u_2$ in $y < 0$. Then u_2 satisfies (3) with $Q = 0$ and, since it has the same z dependence as u_0, must have the form

$$u_2 = A\,\mathrm{e}^{-\omega(\mathrm{i}z-\beta y)/v}$$

in order to behave properly as $y \to -\infty$.

In the conductor let $h_x = u_1$ where, by an analysis similar to that used in deriving (3), u_1 satisfies

$$\frac{\partial^2 u_1}{\partial y^2}+\frac{\partial^2 u_1}{\partial z^2}+\omega^2\left(\frac{1}{c_1^2}-\frac{\mathrm{i}\mu_1\sigma}{\omega}\right)u_1 = 0$$

on account of (2). Here $c_1 = 1/(\mu_1\epsilon_1)^{1/2}$ is the speed of light in the conductor. Giving u_1 the same z dependence as u_2 supplies

$$u_1 = B\,\mathrm{e}^{-\mathrm{i}\omega z/v-\mathrm{i}\zeta y}$$

where

$$\zeta^2 = \omega^2\left(\frac{1}{c_1^2}-\frac{1}{v^2}\right)-\mathrm{i}\sigma\mu_1\omega.$$

By ζ is meant that square root of ζ^2 which has a negative imaginary part. Then u_1 has the appropriate behaviour as $y \to \infty$.

The boundary conditions for u_1, u_2 at the interface $y = 0$ stem from the continuity of E_z and H_x. They are

$$u_0+u_2 = u_1$$

$$\frac{\partial}{\partial y}(u_0+u_2) = \frac{\omega\epsilon}{\omega\epsilon_1-\mathrm{i}\sigma}\frac{\partial u_1}{\partial y}.$$

There results

$$A = [1+(2/\alpha)]\tfrac{1}{2}Q\,\mathrm{e}^{-\omega\beta b/v}$$

$$B = (Q/\alpha)\,\mathrm{e}^{-\omega\beta b/v}$$

where

$$\alpha = \frac{\epsilon\zeta v/\beta}{\sigma+\mathrm{i}\omega\epsilon_1}-1.$$

The consequent expressions for the magnetic field caused by the conductor are, from (5),

$$H_x = \operatorname{Re}\frac{Q}{2\pi}\int_0^\infty \left(1+\frac{2}{\alpha}\right) e^{i\omega(t-z/v)+\omega\beta(y-b)/v}\, d\omega \qquad (y<0)$$

$$= \operatorname{Re}\frac{Q}{\pi}\int_0^\infty \frac{1}{\alpha} e^{i\omega(t-z/v)-i\zeta y-\omega\beta b/v}\, d\omega \qquad (y>0)$$

and the electric field can be deduced in a straightforward manner.

It is certainly not immediately evident from these formulae that the magnetic field in the conductor is independent of the resistivity so they will be examined in more detail in the next section.

3. The distant field

Note first that α cannot vanish for real ω if $\sigma \neq 0$, which will always be assumed. Therefore, there is no difficulty of interpretation with the integrals.

Next an investigation of the field as $y \to \infty$ will be sufficient to tell us whether there are any terms which are not exponentially damped and hence not in accord with the behaviour of skin effect. Therefore, we shall consider the asymptotic form as $y \to \infty$.

It may be helpful if the integral for H_x in $y > 0$ is cast into non-dimensional form by replacing ω by $v\tau/b$. Then

$$H_x = \operatorname{Re}\frac{Qv}{\pi b}\int_0^\infty \frac{1}{\alpha} e^{i\tau(T-Z)-i\eta Y-\beta\tau}\, d\tau \tag{6}$$

where

$$T = vt/b \qquad Z = z/b \qquad Y = y/b \qquad \theta = \sigma\mu_1 vb$$

$$\eta^2 = \tau^2(v^2/c_1^2-1)-i\theta\tau = -\tau^2\beta_1^2 - i\theta\tau.$$

Attention will first be concentrated on the case when $\beta_1^2 > 0$, ie $v < c_1$ so that the charge is travelling at less than the speed of light in the conductor. Let τ_0 be fixed and define the positive δ by

$$2\delta^2 = (\tau_0^2\beta_1^4+\theta^2)^{1/2}+\tau_0\beta_1^2.$$

Then $\operatorname{Re}(i\eta) \geqslant \delta\tau^{1/2}$ for $\tau \geqslant \tau_0$. Hence

$$\left|\int_{\tau_0}^\infty \frac{1}{\alpha} e^{i\tau(T-Z)-i\eta Y-\beta\tau}\, d\tau\right| \leqslant K\int_{\tau_0}^\infty e^{-\beta\tau-\delta Y\tau^{1/2}}\, d\tau,$$

K being some finite constant. Consequently

$$\left|\int_{\tau_0}^\infty \frac{1}{\alpha} e^{i\tau(T-Z)-i\eta Y-\beta\tau}\, d\tau\right| \leqslant K\, e^{-\beta\tau_0-\frac{1}{2}\delta Y\tau_0^{1/2}}.$$

Therefore the integral over $\tau \geqslant \tau_0$ produces a term which decays exponentially as $Y \to \infty$ so long as δ is nonzero. However, δ cannot be zero if at least one of β_1 and θ is nonzero, which will certainly be true provided that the conductivity does not vanish.

Since exponentially small fields are of no concern in this investigation they will be neglected. Accordingly, as $Y \to \infty$ we can write

$$H_x = \mathrm{Re}\frac{Qv}{\pi b}\int_0^{\tau_0}\frac{1}{\alpha}\,\mathrm{e}^{\mathrm{i}\tau(T-Z)-\mathrm{i}\eta Y-\beta\tau}\,\mathrm{d}\tau. \tag{7}$$

Now suppose that τ_0 has been chosen so that $\tau_0 \ll \theta$, which is clearly permissible unless $v \to 0$. Then both α and η can be expanded about $\tau = 0$ with the result that

$$H_x = -\mathrm{Re}\frac{Qv}{\pi b}\int_0^{\tau_0}\left[1+\frac{\epsilon v\theta^{1/2}}{\sigma\beta b}\tau^{1/2}\,\mathrm{e}^{-\pi\mathrm{i}/4}+\mathrm{O}\left(\frac{Y\tau^{3/2}}{\theta^{1/2}}\right)\right]\mathrm{e}^{\mathrm{i}\tau(T-Z)-\nu\tau^{1/2}-\beta\tau}\,\mathrm{d}\tau \tag{8}$$

where $\nu = \theta^{1/2}Y\mathrm{e}^{\pi\mathrm{i}/4}$.

To estimate the integrals in (8) consider

$$I_n = \int_0^{\tau_0}\tau^{n/2}\,\mathrm{e}^{\mathrm{i}\tau(T-Z)-\nu\tau^{1/2}-\beta\tau}\,\mathrm{d}\tau$$

where n is a non-negative integer. If the interval of integration be extended to infinity the error produced is certainly $\mathrm{O}(\mathrm{e}^{-\nu\tau_0^{1/2}})$. Such an error should be neglected as $Y \to \infty$ for consistency with the derivation of (7) where exponentially damped terms were omitted. Therefore

$$I_n = \int_0^{\infty}\tau^{n/2}\,\mathrm{e}^{\mathrm{i}\tau(T-Z)-\nu\tau^{1/2}-\beta\tau}\,\mathrm{d}\tau$$

to our order of accuracy.

Let $\beta-\mathrm{i}(T-Z) = \rho\,\mathrm{e}^{-\mathrm{i}\phi}$ where ρ is positive and $-\frac{1}{2}\pi < \phi < \frac{1}{2}\pi$. Make the substitution $\tau = w^2$ and then deform the contour of integration in the w plane into the radial line making an angle $\frac{1}{2}\phi$ with the real axis. There results

$$I_n = 2\,\mathrm{e}^{\mathrm{i}\phi(n+2)/2}\int_0^{\infty}w^{n+1}\,\mathrm{e}^{-\rho w^2-\nu w\mathrm{e}^{\mathrm{i}\phi/2}}\,\mathrm{d}w.$$

Put $a = \frac{1}{2}\nu\,\mathrm{e}^{\mathrm{i}\phi/2}/\rho^{1/2}$ and change the variable of integration via $w = (\lambda-a)/\rho^{1/2}$.

Then

$$I_n = 2\left(\frac{2a}{\nu}\right)^{n+2}\mathrm{e}^{a^2}\int_a^{\infty}(\lambda-a)^{n+1}\,\mathrm{e}^{-\lambda^2}\,\mathrm{d}\lambda.$$

It is a standard result (see, for example, Abramowitz and Stegun 1965) that as $|a| \to \infty$ with $|\arg a| < \frac{3}{4}\pi$,

$$I_n \sim (n+1)!\frac{2}{\nu^{n+2}}\left[1+\mathrm{O}\left(\frac{1}{a^2}\right)\right]. \tag{9}$$

The conditions on a are met as $Y \to \infty$ since $\arg a = \frac{1}{4}\pi+\phi$.

Employing (9) in (8) with $n = 0$, 1 and 3 we obtain

$$H_x = -\mathrm{Re}\frac{2Qv}{\pi b}\left[\frac{1}{\nu^2}+\frac{2\epsilon v\theta^{1/2}\,\mathrm{e}^{-\pi\mathrm{i}/4}}{\sigma\beta b\nu^3}+\mathrm{O}\left(\frac{1}{Y^4}\right)\right],$$

the estimate for the third term in (8), being valid because putting $T = Z$ ($\phi = 0$) and replacing ν by $|\nu|/\sqrt{2}$ does not affect the rate in (9) at which I_3 tends to zero as $Y \to \infty$.

Hence

$$H_x = \frac{4Qv\epsilon}{\pi\sigma^2\beta\mu_1 y^3} + O\left(\frac{1}{Y^4}\right) \tag{10}$$

as $Y \to \infty$. Exponentially attenuated terms have been ignored and could in any case be regarded as included in the order term of (10).

It is evident from formula (10) that verification has been achieved of Boyer's statement that the magnetic field in the conductor falls algebraically rather than exponentially so that skin effect is not the dominant phenomenon. However, there is disagreement on the actual rate of decay and, moreover, the field is dependent on the conductivity contrary to Boyer's prediction that the magnetic intensity does not involve the resistivity. The reason for the discrepancy is that Boyer's analysis refers to the limiting case $v \to 0$ whereas that possibility is excluded in our derivation of (8). If $v \to 0$ so that $\theta \to 0$, (7) continues to hold but (8) is replaced by

$$H_x = -\mathrm{Re}\frac{Qv}{\pi b}\int_0^{\tau_0} e^{i\tau(T-Z)-Y\tau-\tau}\, d\tau$$

when all terms which have quadratic or higher powers of v are omitted. Hence

$$\begin{aligned} H_x &= -\mathrm{Re}\frac{Qv}{\pi b}[Y - i(T-Z)]^{-1} \\ &= -\frac{Qv}{\pi}\frac{y+b}{(y+b)^2+(vt-z)^2}, \end{aligned} \tag{11}$$

exponentially diminishing terms being again neglected. Formula (11) effectively agrees with that given by Boyer.

It is, however, important to realize that (11) is derived as $v \to 0$ with other quantities held fixed, whereas (10) is relevant for $y/b \to \infty$ while other entities remain constant. If for instance we consider what happens as $\sigma \to \infty$ so that the conductor becomes perfect, (11) indicates that there will be a magnetic field in the conductor but (10) says there will be none (even the order term vanishes because $a \to \infty$). This is because in the former case $\theta \to 0$ while in the latter $\theta \to \infty$. Of course $\theta \to 0$ as $v \to 0$ and $\sigma \to \infty$ can be attained only if $\sigma \to \infty$ slower than v approaches zero—a rather tight restriction. In general (10) would seem to be the better formula to follow because of the constricted range of validity of (11).

4. Čerenkov radiation

The magnitude of the magnetic intensity in the conductor contains the conductivity in the denominator according to (10) and so may be rather small for a good conductor. It may be raised by increasing v. But then the possibility that $v > c_1$ occurs and with it the potentiality for Čerenkov radiation in the conductor (see, for example, Jones 1964). It is of interest to check whether this has any significant influence on the field. Therefore, we now consider the case where $c > v > c_1$.

Equation (6) is still valid but now β_1^2 is negative. Let $\beta_1^2 = -\kappa_1^2$, κ_1 being positive. Then for $\tau \geqslant \tau_0$

$$\mathrm{Re}(i\eta) \geqslant \tau_0^{1/2}[(\tau_0^2\kappa_1^4+\theta^2)^{1/2} - \tau_0\kappa_1^2]^{1/2}$$

and so exponential attenuation again arises for $\tau \geqslant \tau_0$. Thus (7) and hence (10) continues to hold, at any rate so long as β does not become too small to invalidate the asymptotic argument. Consequently there is no change to the formula for the field in the conductor whether or not Čerenkov radiation might take place, as far as a non-relativistic treatment is concerned.

Suppose now that an even larger speed is contemplated so that $v > c$ and Čerenkov radiation might be involved in both media. In that case the formulae at the end of § 2 are still satisfactory provided that β is altered to $\mathrm{i}\kappa$ where κ is real and positive. Bearing in mind (4) we obtain for the field in the conductor:

$$H_x = \frac{Qv}{2\pi b}\int_{-\infty}^{\infty} \frac{1}{\alpha}\,\mathrm{e}^{\mathrm{i}\tau(T-Z)-\mathrm{i}\eta Y-\mathrm{i}\kappa\tau}\,\mathrm{d}\tau$$

where now the path of integration passes below the branch line joining the origin and $\mathrm{i}\theta/\kappa_1^2$, η being defined to be positive as $\tau \to \infty$.

The manipulation is somewhat simplified by writing

$$H_x = \left(\frac{\partial}{\partial T}+\frac{1}{2}\frac{\theta}{\kappa_1^2}\right)\frac{Qv}{2\pi\mathrm{i}b}\int_{-\infty}^{\infty}\frac{\mathrm{e}^{\mathrm{i}\tau(T-Z)-\mathrm{i}\eta Y-\mathrm{i}\kappa\tau}}{\alpha(\tau-\frac{1}{2}\mathrm{i}\theta/\kappa_1^2)}\,\mathrm{d}\tau.$$

It is evident that, since the only zero of α in the complex τ plane lies on the positive imaginary axis, no field is present in the conductor until

$$T > Z+\kappa_1 Y+\kappa, \tag{12}$$

which is to be expected since the wave system is spearheaded by the charge.

Change the variable of integration by $\tau = (\frac{1}{2}\mathrm{i}\theta/\kappa_1^2)(w+1)$ and then

$$H_x = \left(\frac{\partial}{\partial T}+d\right)\frac{Qv}{2\pi\mathrm{i}b}\int_{-1+\mathrm{i}\infty}^{-1-\mathrm{i}\infty}\mathrm{e}^{-d(T-Z-\kappa)(w+1)+\kappa_1 Y d(w^2-1)^{1/2}}\frac{\mathrm{d}w}{\alpha w} \tag{13}$$

where $d = \frac{1}{2}\theta/\kappa_1^2$, $(w^2-1)^{1/2}$ is positive as $w \to \infty$, and

$$\alpha = \frac{(w^2-1)^{1/2}}{a_0-b_0 w}-1 \qquad b_0 = \frac{\epsilon_1\kappa}{\epsilon\kappa_1} \qquad a_0 = \frac{\sigma\kappa b}{\epsilon\kappa_1 v d}-b_0.$$

When (12) is satisfied the path of integration can be deformed into a closed contour C enclosing the branch line and any poles. Applying the differential operator we obtain

$$H_x = -\frac{QvHd}{2\pi\mathrm{i}b}\int_C \mathrm{e}^{-d(T-Z-\kappa)(w+1)+\kappa_1 Y d(w^2-1)^{1/2}}\frac{\mathrm{d}w}{\alpha}$$
$$+\delta(T-Z-\kappa_1 Y-\kappa)\frac{Qv}{2\pi\mathrm{i}b}\int_C \mathrm{e}^{-d(T-Z-\kappa)(w+1)+\kappa_1 Y d(w^2-1)^{1/2}}\frac{\mathrm{d}w}{\alpha w}, \tag{14}$$

H being the Heaviside unit function which is unity when (12) holds and zero when the reverse inequality is valid. Deform the contour C into the ellipse $w = \cos(\mathrm{i}\lambda-\psi)$ where the positive λ is defined by

$$\cosh\lambda = (T-Z-\kappa)/[(T-Z-\kappa)^2-\kappa_1^2 Y^2]^{1/2}.$$

The exponent in the integrals becomes

$$-d(T-Z-\kappa)-d[(T-Z-\kappa)^2-\kappa_1^2 Y^2]^{1/2}\cos\psi$$

and to each integral is added a contribution from the pole due to the zero $w = w_0$ of α if $\cosh \lambda < w_0$.

In the second integral of (14) we are concerned only with $T = Z + \kappa + \kappa_1 Y$ and then λ becomes infinite. There is no contribution from the pole and the integral may be evaluated easily. Hence the part of H_x stemming from this term is

$$-\delta(T - Z - \kappa - \kappa_1 Y) \frac{Qvb_0}{b(1+b_0)} \mathrm{e}^{-\kappa_1 Yd}.$$

As regards the first integral of (14) it is evident from the above analysis that it does not exceed a term of order

$$\exp\{-d(T-Z-\kappa) + d[(T-Z-\kappa)^2 - \kappa_1^2 Y^2]^{1/2}\}.$$

Therefore, at a fixed location in space the magnetic field decreases exponentially as time increases after the passage of the wavefront. Consequently the field can be regarded as exponentially small except in a neighbourhood of the wavefront. Near the wavefront the first term of (14), by a similar analysis to that for the second, contributes

$$-\frac{QvHda_0}{b(1+b_0)^2} \mathrm{e}^{-\kappa_1 Yd}.$$

Thus the wavefront part of the magnetic intensity is given by

$$H_x = -\frac{Qv\,\mathrm{e}^{-\kappa_1 Yd}}{b(1+b_0)} \left(b_0 \delta(T - Z - \kappa - \kappa_1 Y) + \frac{Hda_0}{1+b_0} \right).$$

and even this exhibits exponential decay with increasing distance from the interface.

It may therefore be concluded that when the charge is travelling faster than the speed of light in either medium, the magnetic intensity falls off exponentially in the conductor as compared with its value at the boundary and also with distance behind the wavefront. This is not surprising because the energy of the incident wave is concentrated on the arms of the Mach wedge and so its spectrum is mainly in the high frequencies. Thus skin effect will be a significant phenomenon and one can expect the best concentration of energy in the conductor to be near the surface.

Broadly speaking, then, non-relativistic classical electromagnetic theory predicts that the largest magnetic intensity occurs in the conductor when the charge is moving at less than the speed of light in the incident medium and, in so far as (10) is valid for the entire range, when its speed is the highest possible consistent with this condition.

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